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GROUP TESTING WITH TWO DEFECTIVES

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Recently the problem of determining the minimax number of group tests for finding two defectives separately contained in two disjoint sets has been completely solved. However, the closely related problem of determining the minimax number of group tests for finding two defectives contained in one set remains open. This is a surprisingly difficult combinatorial problem with very little known. In this paper we give a partial solution to this problem.

1. Introduction

Consider a population of n items consisting of d *defectives* and $n-d$ *good* ones. The problem is to find these d defectives by means of a sequence of group tests. We will call a set of items *contaminated* if it contains at least one defective, and *pure* if otherwise. Then a *group test* is a test on a given set with two possible outcomes: The set is either identified as a contaminated set or as a pure set. let $M_g(d,n)$ be the maximum number of tests the group testing procedure g requires to find the d defectives in n items. Define

$$M(d,n)=\min_g M_g(d,n).$$

Then $M(d,n)$ is the minimax test number for given d and n.

A halving procedure is one which always tests half (or as close as half) of a contaminated set. Therefore if we start with n items containing at least one defective, then the halving procedure will find one defective in $\lceil \log_2 n \rceil$ tests where $\lceil x \rceil$ denotes the smallest integer but not smaller than x. It is well known that the halving procedure yields the minimax number for $d=1$, hence $M(1,n)=\lceil \log_2 n \rceil$. However, the determination of $M(2,n)$ is a surprisingly difficult combinatorial problem and very little is known except some recent results [1,2] on the special case that the two defectives have been separately contained in two disjoint sets and a paper by Sobel [5] concerning the expected number of tests. In this paper we give a partial

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solution to the $M(2,n)$ problem. More specifically, let n_k denote the largest n such that $M(2,n)\le k$. Then we determine an upperbound u_k and a lowerbound l_k for n_k such that

$$l_k/u_k>0.95.$$

We show that this is a significant improvement over previously known results.

2. An upperbound for n_k

For $k\ge 1$, let I_k denote the integer such that

$$\binom{I_k}{2}<2^k<\binom{I_k+1}{2}.$$

Since no integer x can be a solution to the equation $\binom{x}{2}=2^k$ for $k\ge 1$, no ambiguity arises in our definition of I_k. At the beginning, the two defectives in the n items can be any of the $\binom{n}{2}$ possible pairs. Since each test has binary outcomes, the determination of the two defectives needs at least $\lceil \log_2\binom{n}{2}\rceil$ tests. Therefore I_k, usually known as the *information-theoretic upperbound,* is clearly an upperbound for n_k. Define $u_k=I_k-1$. We show that for $k\ge 4$, u_k is also an upperbound for n_k. The following two lemmas demonstrate some useful properties of u_k.

Lemma 1. $u_k=\lfloor 2^{(k+1)/2}-\frac{1}{2}\rfloor$ *where* $\lfloor x\rfloor$ *denotes the largest integer not exceeding* x.

Proof. It suffices to show that

$$\binom{u_k+1}{2}<2^k<\binom{u_k+2}{2}.$$

Since

$$u_k+1\le 2^{(k+1)/2}+\tfrac{1}{2}<u_k+2,$$

it follows that

$$\binom{u_k+1}{2}\le\tfrac{1}{2}(2^{(k+1)/2}+\tfrac{1}{2})(2^{(k+1)/2}-\tfrac{1}{2})=2^k-\tfrac{1}{8}<\binom{u_k+2}{2}$$

or

$$\binom{u_k+1}{2}<2^k<\binom{u_k+2}{2}.$$

Lemma 2.

$$\binom{u_{k+1}+1}{2}-\binom{u_k}{2}>2^k.$$

Proof. For k even, $u_{k+1}+1=2^{(k+2)/2}$. Therefore

$$\binom{u_{k+1}+1}{2}-\binom{u_k}{2}=\tfrac{1}{2}\{2^{(k+2)/2}(2^{(k+2)/2}-1)-u_k(u_k-1)\}$$
$$=\tfrac{1}{2}\{2^{k+2}-2^{(k+2)/2}-u_k(u_k+1)+2u_k\}$$
$$=\left\{2^{k+1}-\binom{u_k+1}{2}\right\}-\{u_k-2^{k/2}\}>2^k,$$

since

$$2^{k+1}-\binom{u_k+1}{2}>2^{k+1}-2^k=2^k$$

and

$$\binom{2^{k/2}+1}{2}<2^k \quad \text{implies} \quad u_k\geq 2^{k/2}.$$

For k odd, $u_k+1=2^{(k+1)/2}$. Therefore

$$\binom{u_{k+1}+1}{2}-\binom{u_k}{2}=\tfrac{1}{2}\{u_{k+1}(u_{k+1}+1)-(2^{(k+1)/2}-1)(2^{(k+1)/2}-2)\}$$
$$=\tfrac{1}{2}\{(u_{k+1}+2)(u_{k+1}+1)-2u_{k+1}-2^{k+1}+3\cdot 2^{(k+1)/2}-4\}$$
$$=\left\{\binom{u_{k+1}+2}{2}-2^k\right\}+\{3\cdot 2^{(k-1)/2}-(u_{k+1}+2)\}>2^k,$$

since

$$\binom{u_{k+1}+2}{2}-2^k>2^{k+1}-2^k=2^k$$

and

$$\binom{3\cdot 2^{(k-1)/2}}{2}>2^{k+1} \quad \text{implies} \quad 3\cdot 2^{(k-1)/2}\geq u_{k+1}+2.$$

Theorem 1. *$n_k\leq u_k$ for $k\geq 4$.*

Proof. It is easy to verify that $n_4=5=u_4$. We prove Theorem 1 by induction on k. We show that $M(2,u_k+1)>k$ for any arbitrary procedure g.

Suppose the first test of g is on a set of cardinality x. If $x<u_k+1-u_{k-1}$, consider the outcome that the set is pure. Then the $(2,u_k+1)$ problem is reduced to the $(2,u_k+1-x)$ problem. Since $u_k+1-x>u_{k-1}$, we need at least k more tests by the induction assumption. If $x\geq u_k+1-u_{k-1}$, consider the outcome that the set is contaminated. The number of possible specifications of the two defectives after the first test is

$$\binom{u_k+1}{2}-\binom{u_k+1-x}{2}\geq\binom{u_k+1}{2}-\binom{u_{k-1}}{2}>2^{k-1}\quad \text{by Lemma 2.}$$

Therefore again, at least k more tests are needed.

3. A lowerbound for n_k

Define $l_1=2$, $l_2=3$, $l_3=4$ and for $k\geq 4$,

$$l_k=\begin{cases}\lceil 43\cdot 2^{k/2-5}\rceil-1 & \text{for } k \text{ even,}\\ \lceil 31\cdot 2^{(k-1)/2-4}\rceil-1 & \text{for } k \text{ odd.}\end{cases}$$

We will prove that l_k is a lowerbound for n_k. First we quote a result which we will make frequent uses in our proof.

Theorem 2 [2]. *Suppose the two defectives are contained separately in two disjoint sets with cardinalities x and y respectively. Then $\lceil \log_2 xy\rceil$ tests suffice to find the two defectives.*

Now we state and prove our main result.

Theorem 3. $n_k\geq l_k$.

Proof. It is easy to verify Theorem 3 for $k<6$. For $k\geq 6$, we prove Theorem 3 by demonstrating a procedure g such that $M_g(2,l_k)\leq k$.

Case (i). *k is even.* We partition the l_k items into four disjoint sets A,B,C and D with cardinalities $a=2^{k/2-3}$, $b=2^{k/2-2}$, $c=2^{k/2-2}$ and $d=\lceil 31\cdot 2^{k/2-5}\rceil-1-2^{k/2-2}$ respectively. It is easily checked that

$$a+b+c+d=l_k \quad\text{and}\quad c+d=l_{k-1}.$$

For the procedure g, we first test the set $A\cup B$. In the case that $A\cup B$ is pure, we are left with l_{k-1} items containing two defectives which can be found in $k-1$ more tests by induction. In the case that $A\cup B$ is contaminated, we next test the set $A\cup C$. If $A\cup C$ is pure, then B must be a contaminated set. We use the halving procedure to find one defective in B in $\frac{1}{2}k-2$ test and we are left with at most $b+d-1=\lceil 31\cdot 2^{k/2-5}\rceil-2<2^{k/2}$ items containing one defective. We find the remaining defective by the halving procedure in $\frac{1}{2}k$ tests and count a total of $2+\frac{1}{2}k-2+\frac{1}{2}k=k$ tests for the whole problem.

If $A\cup C$ is contaminated, we proceed to test the set $B\cup C$. In the case that $B\cup C$ is pure, then A must be a contaminated set. We use the halving procedure to find one defective in A in $\frac{1}{2}k-3$ tests and we are left with at most $a+d-1=\lceil 31\cdot 2^{k/2-5}\rceil-1-2^{k/2-3}<2^{k/2}$ items containing one defective. We find the remaining defective by the halving procedure in $\frac{1}{2}k$ more tests and count a total of $3+\frac{1}{2}k-3+\frac{1}{2}k=k$ tests.

In the case that $B\cup C$ is contaminated, then D must be a pure set. We proceed to test the set A. If A is pure, then necessarily both B and C are contaminated sets. We use the halving procedure to find each defective in $\frac{1}{2}k-2$ tests and count a total of $4+\frac{1}{2}k-2+\frac{1}{2}k-2=k$. If A is contaminated, then we use the halving procedure to find one defective in A in $\frac{1}{2}k-3$ tests and we are left with at most $b+c-1=2^{k/2-2}+2^{k/2-2}-1<2^{k/2-1}$ items containing one defective. We find the remaining defective by the halving procedure in $\frac{1}{2}k-1$ more tests and count a total of $4+\frac{1}{2}k-3+\frac{1}{2}k-1=k$ tests.

Case (ii). *k is odd.* We partition the l_k items into six disjoint set A, B, C, D, E, and F with cardinalities

$$a=2^{(k-1)/2-3}, \qquad b=2^{(k-1)/2-1}-\lfloor 2^{(k-1)/2-5}\rfloor-\delta_9,$$
$$c=2^{(k-1)/2-1}+\lfloor 2^{(k-1)/2-5}\rfloor+\delta_9, \qquad d=2^{(k-1)/2-1}+\lfloor 2^{(k-1)/2-5}\rfloor,$$
$$e=2^{(k-1)/2-2}+\lfloor 2^{(k-1)/2-6}\rfloor \quad \text{and}$$
$$f=\lceil 31\cdot 2^{(k-1)/2-6}\rceil-1-2^{(k-1)/2-1}+\lfloor 2^{(k-1)/2-5}\rfloor+\delta_7+\delta_9,$$

where $\delta_i=1$ for $k=i$ and 0 otherwise. It is easily checked that

$$a+b+c+d+e+f=l_k,$$
$$c+d+e+f=l_{k-1} \quad \text{and} \quad b+f=l_{k-4}.$$

For the procedure g, we first test the set $A\cup B$. In the case that $A\cup B$ is pure we are left with $c+d+e+f=l_{k+1}$ items containing two defectives which can be found in $k-1$ more tests by induction. In the case that $A\cup B$ is contaminated, we next test the set $A\cup C$. If $A\cup C$ is pure, then B must be contaminated; we test the set D. If D is contaminated, we can find the two defectives in B and D in $\lceil \log_2 bd\rceil \le k-3$ tests by Theorem 2 and count a total of $3+k-3=k$ tests. In the case that D is pure, we proceed to test the set E. If E is contaminated, again by using Theorem 2 we can find the two defectives in B and E in $\lceil \log_2 be\rceil \le k-4$ tests and count a total of $4+k-4=k$ tests. If E is pure, we are left with $b+f=l_{k-4}$ items containing two defectives which can be found in $k-4$ more tests by induction. The total number of tests is again $4+k-4=k$.

If $A\cup C$ is contaminated, we proceed to test the set A. In the case that A is pure, then both B and C must be contaminated. By Theorem 2 we can find the two defectives in $\lceil \log_2 bc\rceil \le k-3$ tests and count a total of $3+k-3=k$ tests. In the case that A is contaminated, we use the halving procedure to find one defective in A in $\frac{1}{2}(k-1)-3$ tests and we are left with at most

$$a+b+c+d+e+f-1=[31\cdot 2^{(k-1)/2-4}]-2<2^{(k-1)/2+1}$$

items containing one defective. We find the remaining defective by the halving procedure in $\frac{1}{2}(k-1)+1$ more tests and count a total of $3+\frac{1}{2}(k-1)-3+\frac{1}{2}(k-1)+1=k$ tests. The proof is complete.

Theorem 4. $l_k/u_k > 0.95$.

The proof is straightforward by using Lemma 1, the definition of l_k and direct calculations for $k \le 9$.

4. Some concluding remarks

For $4 \le k \le 7$, $n_k = u_k = l_k$. For $8 \le k \le 11$, it is known that $n_k = u_k > l_k$. For $k = 12$, $n_k = 89 < u_k = 90$ and we suspect that n_k and u_k diverge from then on.

It is observed in [3] that any merging procedure for two strings with lengths 2 and $n-2$ can be converted to a group testing procedure with n items and 2 defectives. In particular, the minimax merging procedure given in [4] can be converted to a group testing procedure which yields the following lowerbound m_k:

$$m_k = \begin{cases} \lfloor \frac{17}{7} 2^{k/2-1} \rfloor + 2 & \text{for } k \text{ even,} \\ \lfloor \frac{12}{7} 2^{(k-1)/2} \rfloor + 2 & \text{for } k \text{ odd.} \end{cases}$$

The ratio of m_k and u_k is about 0.85. Therefore the new lowerbound l_k improves over m_k by a constant factor.

After this paper was submitted for publication, a paper of Tošić [6] appeared recently in which a lowerbound t_k for n_k was given as:

$$t_k = 2^{k/2} + F_{k/2} \qquad \text{for } k \text{ even,}$$

$$t_k = 3 \cdot 2^{(k-1)/2} + F_{(k-1)/2} \qquad \text{for } k \text{ odd,}$$

where F_k is the kth Fibonacci number (i.e., $F_1 = F_2 = 1$, $F_k = F_{k-1} + F_{k-2}$ for $k \ge 3$). It is easily seen that for k large, t_k/u_k tends to $1/\sqrt{2} = 0.707\cdots$ for k even and to $0.75\cdots$ for k odd. Therefore t_k is not an improvement over m_k (in fact it can be shown $l_k \ge m_k \ge t_k$ for every k).

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